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Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 69 (2005) 844–851

PERIODIC BILLIARD TRAJECTORIES IN A MAGNETIC FIELD†

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(*Received* 27 *January* 2004)

The problem of the existence of periodic trajectories of a charged particle in a magnetic field, when the particle moves inside a closed convex region and is elastically reflected from its boundary, is considered. The presence of an infinite number of different periodic trajectories at low magnetic field strengths is established using Poincaré's geometric theorem. The conditions for two-link trajectories to be stable in the case of a uniform magnetic field are obtained.

1. A BILLIARD IN A MAGNETIC FIELD

Consider the motion of a charged particle of unit mass in a magnetic field. We will assume that the lines of induction of the magnetic field are perpendicular to a certain plane π in which the particle is situated, and its velocity is directed along π. Then, the particle will remain in the plane π during the whole time it is moving. Its motion is described by a Hamiltonian dynamical system with the Hamilton function

$$H = (\upsilon_\xi^2 + \upsilon_\eta^2)/2$$

and the symplectic structure

$$\omega = d\upsilon_\xi \wedge d\xi + d\upsilon_\eta \wedge d\eta - eBc^{-1}d\xi \wedge d\eta$$

where ξ, η are the Cartesian coordinates of the particle in the plane, υ_ξ, υ_η are the components of the velocity, e is the charge of the particle, c is the velocity of light, $B(\xi, \eta)$ is the magnetic induction and $\wedge$ is the outer product.

An important characteristic of the motion is the Larmor radius, a quantity which is calculated from the formula

$$R = R(\xi, \eta) = \upsilon c/(e|B(\xi, \eta)|) \tag{1.1}$$

where υ is the particle velocity.

We will assume that the particle remains inside a region bounded by the convex curve L throughout its motion, and is reflected on collision with L according to the law of a perfectly elastic impact (the velocity component tangent to L is conserved while the normal component changes sign).

The magnetic induction is a unique function of two variables ξ, η, specified in advance. We will call this dynamical system with impacts a unilateral billiard. It has been investigated by many researchers (see, for example, the review [1]).

A unilateral billiard in a circle is a completely integrable system (in addition to the energy integral there is also an integral that is linear in the velocities). However, if we take as the boundary L an ellipse with unequal semi-axes, the corresponding system will not allow of an additional analytic first integral [2].

†*Prikl. Mat. Mekh.* Vol. 69, No. 6, pp. 942–949, 2005.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.11.006

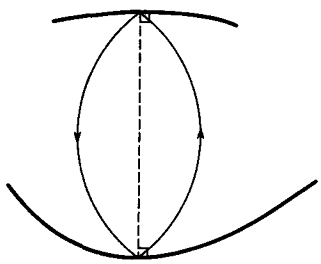

Fig. 1

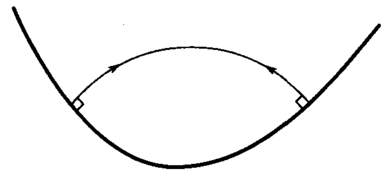

Fig. 2

A case of particular interest is when the direction of the magnetic field (the sign of the induction) reverses at each collision of the particle with the curve L. This system will be called a bilateral billiard.

The term bilateral corresponds to the representation of a billiard as the motion of a particle along an extremely solid packed surface σ on which a magnetic field, directed out of (or into) σ acts. For example, one can take as σ the surface of an ellipsoid, one of the semi-axes of which approaches zero. In fact, a magnetic field without singularities cannot always be directed inward or outward to a closed surface σ, since its flux through σ is always equal to zero. However, in non-classical electromagnetic theories this is quite possible (for example, Dirac's monopole theory). A bilateral billiard gives a simple (true, a degenerate) model of a magnetic monopole.

Below we consider the problem of periodic closed trajectories of billiards in a magnetic field.

We will characterize the position of a particle on the curve L by two quantities: s – a parameter along the curve L, proportional to the natural parameter, and γ the sine of the angle between the inward normal to L and the vector of the particle velocity immediately after recoil – the angle of reflection (compare with [3]).

Following the approach described previously in [4], we determine the sequence of points $s^1, s^2, \dots$, s^n, which specify the periodic n-link trajectory of the billiard (the parameter s^i corresponds to the ith point of collision of the particle with L). Suppose $s^2-s^1, s^3-s^2, \dots, s^n-s^{n-1}, s_*^1-s^n$ are included between 0 and 2π, where $s_*^1 - s^1 \mathrm{mod} 2\pi$ (s_*^1 is the coordinate of the closure point of the trajectory).

We will say that the trajectory of the billiard performs k rotations about the boundary L if $s_*^1 - s^1 = 2\pi k$. The number k defines the geometrical types of closed trajectories. Clearly, a two-link trajectory of a billiard (when $n = 2$) performs a single rotation about the curve L.

Henceforth it will be of particular interest to represent the case when the magnetic field is uniform, i.e. its induction is independent of the position of the particle in the plane. In this case, between two collisions with the boundary, the particle moves along the arc of a circle of Larmor radius (1.1), which is also independent of the position of the particle.

The two-link periodic trajectory of a unilateral billiard in a uniform magnetic field consists of two arcs, symmetrically positioned about their contracting chord. This chord must obviously be perpendicular to the curve L at points of intersection (Fig. 1).

In a bilateral billiard, the two-link trajectory is an arc of a circle of Larmor radius, which is perpendicular to the curve L at points of intersection. The charged particle traverses this arc in both directions (Fig. 2).

2. THE CONDITIONS FOR THE EXISTENCE OF PERIODIC TRAJECTORIES

We will fix a constant of the energy integral: H = const > 0. In the phase space of this system with elastic impacts, the level of the energy D is a three-dimensional manifold with a boundary. Its boundary ∂D is a ring, corresponding to the position of the particle at the boundary L. We will consider the general case when the magnetic field is non-uniform. We will assume that its induction B is a smooth function of the Cartesian coordinates ξ, η.

As previously [5], we can determine the curve L using Frenet's formulae by specifying the initial position of the vectors of the accompanying basis at points of the curve and its curvature. With this specification, the curvature of the curve L will always be positive.

At each point of the closed region, bounded by the curve L, we calculate the Larmor radius (1.1) and we let $R_{\min}$ be the least of the values obtained.

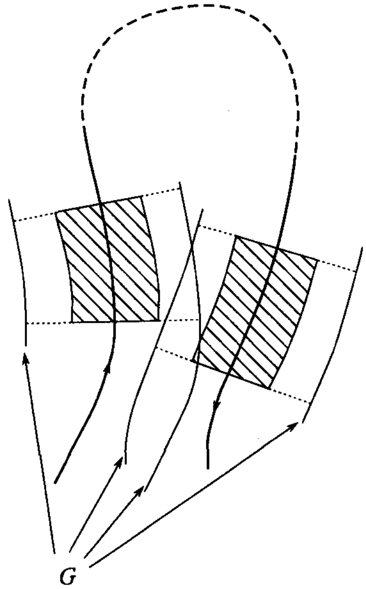

Fig. 3

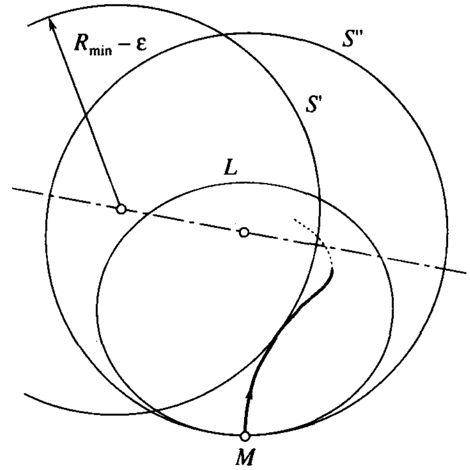

Fig. 4

Lemma. Suppose the radius of curvature of the curve L is always less than R_{min}. Then the trajectory of a particle ejected transversely from a point on the curve L, intersects it again and also transversely.

Proof. We release a particle from a certain point M on the curve L transversely to the boundary. We consider the geometrical position G of the centres of the circles of radius $R_{min} - \varepsilon$, where ε is fairly small, tangent to the trajectory of the particle. According to the condition of the lemma, G is formed by two smooth curves [1] and forms the boundary of a curvilinear strip of width $2(R_{min} - \varepsilon)$, the axis (the median line) of which is the trajectory of the particle (Fig. 3).

The strip considered can have a self-intersection, but under the conditions of the lemma the trajectory of the particle cannot intersect its boundary. We will show this.

For a certain position of the particle on the trajectory we draw a circle S' of radius $R_{min} - \varepsilon$, touching the trajectory (Fig. 4). Note that the centre of the circle S' belongs to G. We also draw a circle S'' of the same radius, as that of S', also touching the curve L at the point M. By the condition of the lemma the curve L only intersects the circle S'' once [1]. The trajectory of the particle also intersects the circle S' only once. In the opposite case one of the samples of a section of the trajectory between the intersections with S' (when we shift it along the straight line connecting the centres of S' and S'') touches S'' from the inside.

Hence, when the particle moves, the area of that part of the plane π which the particle cannot enter (this is the strip which "is covered" by the section of length $R_{min} - \varepsilon$ in the middle of which the particle exists, while the section moves together with the particle along the trajectory, remaining perpendicular to the trajectory), is increased with a velocity of not less than $\upsilon(R_{min} - \varepsilon)$, whereas the region of motion of the particle is bounded by the curve L.

Thus, being released transverse to the curve L, the particle should ultimately again appear on L, where tangent contact with the curve L is excluded by the condition of the lemma, which was also required.

The main result is as follows.

Theorem 1. Under the conditions of the lemma, for any $n > 1$ and any $k < n$, relatively prime with n, at least two different n-link periodic trajectories of a unilateral billiard exist, which perform k rotations about the boundary L. In the case of a bilateral billiard, the assertion of the theorem remains true for any even n.

In the case of a uniform magnetic field (B = const), a weaker version of Theorem 1 was stated in [1]: it is asserted that under the conditions of the theorem for $n > 2$ there is at least one periodic n-link trajectory of a unilateral billiard. It corresponds to the maximum point of a certain function of n variables (which corresponds to the Jacobi action on a certain class of closed curves) on a compact manifold with a piecewise-smooth boundary. However, the arguments presented in [1] cannot be regarded as rigorous. A complete variational proof of Theorem 1 was given in [6] for the case when there is no magnetic field, and it uses non-trivial topological ideas.

In the general case, when the magnetic field is non-uniform, one can use Novikov's variational theory, which touches on the existence of closed trajectories of irreversible systems with a compact two-dimensional configurational manifold (in our case this manifold is homeomorphic with a two-dimensional sphere) [7–9]. We will state the sufficient condition for at least one closed trajectory of a particle of unit mass to exist in a uniform magnetic field, which Novikov's theory gives

$$\lambda v < eBc^{-1}\sigma \tag{2.1}$$

Here λ is the length of the boundary L and σ is the area enclosed inside L. Condition (2.1) applies both for a unilateral and for a bilateral billiard. For example, if L is a circle of radius r, inequality (2.1) takes the form $r > 2R$, where R is the Larmor radius (1.1). With these assumptions the condition of Theorem 1 reduces to the inequality $r < R$.

To prove Theorem 1 we use Poincaré's geometric theorem (compare with [3]). We will consider the Poincaré map for the dynamical system T: $\partial D \to \partial D$. In view of the lemma, the map T is continuous and one-to-one. The conservation of the 2-form of the area $d\gamma \wedge ds$ by virtue of the map T follows from the conservation of the 2-form ω in view of the Hamiltonian system of differential equations with Hamiltonian H. Moreover, we note that

$$T:(s,1)\to(s,1) \quad \text{and} \quad T:(s,-1)\to(s+2\pi,-1)$$

It is clear that the map T cannot have invariant points not lying on the boundary of the ring. Hence, we will consider the compound transformation consisting of T^n (i.e. T applied n times) and rotations of the ring R_k around the centre by an angle $-2\pi k$, where $k < n$ and the number k is relatively prime with n. Since R_k is obviously continuous, one-to-one and area conserving transformation of the ring into itself, the compositions R_kT^n satisfy all the conditions of Poincaré's geometric theorem [3, 4] and, consequently, have two series of geometrically different invariant points

$$U, T(U), T^2(U), \ldots, T^{n-1}(U); \quad V, T(V), T^2(V), \ldots, T^{n-1}(V)$$

All these points are invariant under the action of R_kT^n and are rotated by an angle $2\pi k$ due to the action of T^n. Then, to each series of points there corresponds a closed n-link trajectory of the billiard, performing k rotations about the curve L.

Clearly a bilateral billiard can only have closed trajectories with an even number of links. But, to determine the Poincaré map in the case of a bilateral billiard it is necessary to consider its two iterations: T_+, corresponding to the motion of the particle along one side of the surface, and T_-, corresponding to the motion along the reverse side (when the direction of the magnetic induction is changed).

3. CONDITIONS OF STABILITY OF TWO-LINK TRAJECTORIES OF A BILLIARD IN A UNIFORM MAGNETIC FIELD

The problem of the orbital stability of two-link trajectories of a unilateral billiard in a uniform magnetic field (correspondingly, of a bilateral billiard in an "almost" uniform magnetic field) reduces to investigating the stability of a fixed point of the corresponding doubly employed Poincaré map T^2 (the composition $T_- \circ T_+$).

We introduce Cartesian coordinates (ξ, η) in the plane of motion of the particle and we consider the iteration of the Poincaré map. Suppose that, in the neighbourhood of the initial point of the arc of the trajectory between two collisions, the curve L is specified by the equation $\eta = \Phi(\xi)$, and near the final point by the equation $\eta = \Psi(\xi)$.

It is convenient to write the mapping in (ξ, γ) coordinates, where γ is the sine of the angle of reflection.

For Cartesian coordinates of the centre of the Larmor circle (ξ_c, η_c), along the arc of which the particle moves from a collision with the curve Φ to a collision with the curve Ψ, the following formulae hold

$$\begin{aligned}
\xi_c &= \xi_s - \tilde{R}(\xi_s)\left(\sqrt{1-\gamma_s^2} + \gamma_s\Phi'(\xi_s)\right) \\
\eta_c &= \Phi(\xi_s) + \tilde{R}(\xi_s)\left(\gamma_s - \sqrt{1-\gamma_s^2}\,\Phi'(\xi_s)\right) \\
\tilde{R}(\xi_s) &= R/\sqrt{1+(\Phi'(\xi_s))^2}
\end{aligned} \tag{3.1}$$

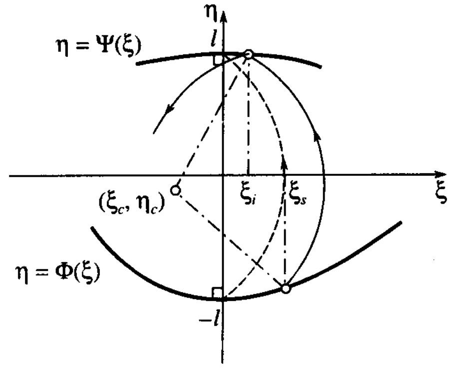

Fig. 5

Here ξ_s, γ_s are coordinates describing the position of the particle on the curve Φ, and R is the Larmor radius.

Iteration of the mapping is specified by the formulae

$$(\xi_i - \xi_c)^2 + (\Psi(\xi_i) - \eta_c)^2 = R^2, \quad \gamma_i = \frac{\eta_c - \Psi(\xi_i) + (\xi_i - \xi_c)\Psi'(\xi_i)}{R\sqrt{1 + (\Psi'(\xi_i))^2}} \tag{3.2}$$

The coordinates (ξ_i, γ_i) describe the position of the particle on the curve Ψ.

Consider the case of a unilateral billiard (Fig. 5). Suppose the two-link periodic trajectory connects points with Cartesian coordinates $(0, -1)$ and $(0, 1)$, and

$$\Phi(\xi) = -l + a_{21}\xi^2/2 + O_3(\xi), \quad \Psi(\xi) = l - a_{22}\xi^2/2 + O_3(\xi)$$

We will also assume that $a_{21} > 0$, $a_{22} > 0$ and $a_{21} > a_{22}$. It is obvious that

$$a_{21} = 1/R_1, \quad a_{22} = 1/R_2$$

where R_1 and R_2 are the radii of curvature of the curves Φ and Ψ at the points with Cartesian coordinates $(0, -1)$ and $(0, 1)$ respectively.

The doubly employed Poincaré map has a fixed point, which in (ξ, γ) coordinates has the form $(0, l/R)$.

We introduce the following notation

$$c_1 = 1 - 2la_{21}, \quad c_2 = 1 - 2la_{22}, \quad d = 2lR/\sqrt{R^2 - l^2}$$

Using formulae (3.1) and (3.2) we obtain, in the linear approximation, the following expressions for (ξ_i, γ_i)

$$\left\| \begin{matrix} \xi_i \\ \gamma_i + \dfrac{l}{R} \end{matrix} \right\| = \left\| \begin{matrix} c_1 & d \\ \dfrac{c_1c_2 - 1}{d} & c_2 \end{matrix} \right\| \left\| \begin{matrix} \xi_s \\ \gamma_s - \dfrac{l}{R} \end{matrix} \right\| + O_2 \tag{3.3}$$

For the next iteration of the map T, which returns the particle to the curve Φ, formulae similar to (3.1) and (3.2) hold. Using them we obtain

$$\left\| \begin{matrix} \xi_f \\ \gamma_f - \dfrac{l}{R} \end{matrix} \right\| = \left\| \begin{matrix} c_2 & d \\ \dfrac{c_1c_2 - 1}{d} & c_1 \end{matrix} \right\| \left\| \begin{matrix} \xi_i \\ \gamma_i + \dfrac{l}{R} \end{matrix} \right\| + O_2 \tag{3.4}$$

Here (ξ_f, γ_f) are the coordinates of the particle, which is once again on the curve Φ.

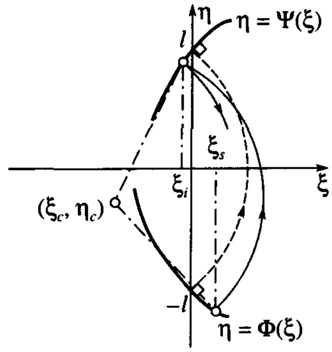

Fig. 6

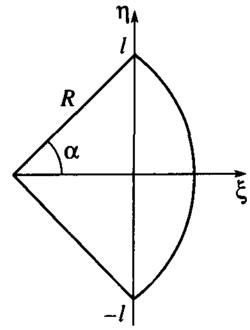

Fig. 7

From formulae (3.3) and (3.4) we obtain the following well-known result [1]: in the case of a unilateral billiard the fixed point $(0, l/R)$ of the doubly employed map T has a hyperbolic type when one of the following conditions is satisfied: $R_1 < 2l < R_2$ or $2l > R_1 + R_2$. If one of the conditions $0 < 2l < R_1$ or $R_2 < 2l < R_1 + R_2$ is satisfied, the fixed point has an elliptic type. It was also noted in [1], that the inequalities obtained agree exactly with the conditions for stability, which hold for a two-link trajectory of the "usual" billiard (for the inertial motion of a particle) (see [4]).

We will now consider the case of a bilateral billiard (Fig. 6). Suppose a two-link periodic trajectory, as before, connects points with the Cartesian coordinates $(0, -1)$ and $(0, 1)$. However, now

$$\Phi(\xi) = -l - a_1\xi + a_{21}\xi^2/2 + O_3(\xi), \quad \Psi(\xi) = l + a_1\xi - a_{22}\xi^2/2 + O_3(\xi)$$

Here $a_1 > 0$. As in the case of a unilateral billiard, we assume that $a_{21} > 0$, $a_{22} > 0$ and $a_{21} > a_{22}$.

Moreover, the following formulae hold

$$\cos^2\alpha = 1/(1 + a_1^2), \quad \sin\alpha = l/R$$

where α is half the angle subtended by the arc of the circle – the two-link periodic trajectory (Fig. 7).

The radii of curvature R_1 and R_2 of the curves Φ and Ψ at the points with Cartesian coordinates $(0, -1)$ and $(0, 1)$ can be calculated from the formulae

$$1/R_1 = a_{21}(a_1^2 + 1)^{-3/2}, \quad 1/R_2 = a_{22}(a_1^2 + 1)^{-3/2}$$

respectively.

The composition $T_- \circ T_+$ has the fixed point $(\xi, \gamma) = (0, 0)$.

We introduce the notation

$$p_1 = \frac{1 - 2la_{21} - a_1^4}{(1 + a_1^2)^2}, \quad p_2 = \frac{1 - 2la_{22} - a_1^4}{(1 + a_1^2)^2}$$

$$q_1 = \frac{2l}{1 + a_1^2}, \quad q_2 = \frac{p_1p_2 - (1 - a_1^4)^2}{q_1} - \frac{2l}{R^2}$$

Iteration of the map T_+: $(\xi_s, \gamma_s) \to (\xi_i, \gamma_i)$ is specified by formulae (3.1) and (3.2). From them we obtain

$$\left\| \begin{matrix} \xi_i \\ \gamma_i \end{matrix} \right\| = \left\| \begin{matrix} p_1 & q_1 \\ q_2 & p_2 \end{matrix} \right\| \left\| \begin{matrix} \xi_s \\ \gamma_s \end{matrix} \right\| + O_2 \tag{3.5}$$

For iteration of the map T_-, which returns the particle to the curve Φ, we have

$$\left\|\begin{matrix}\xi_f\\ \gamma_f\end{matrix}\right\| = \left\|\begin{matrix}p_2 & q_1\\ q_2 & p_1\end{matrix}\right\|\left\|\begin{matrix}\xi_i\\ \gamma_i\end{matrix}\right\| + O_2 \tag{3.6}$$

Here (ξ_f, γ_f) are the coordinates of the particle, once again situated on the curve Φ.
We calculate the trace of the product of the matrices from formulae (3.5) and (3.6)

$$\tau = 2\left(\frac{2(2l\cos\alpha)^2}{R_1R_2} + 2(2l\cos\alpha)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{a_1^2 - 1}{a_1^2 + 1} + \frac{a_1^4 - 6a_1^2 + 1}{(a_1^2 + 1)^2}\right) \tag{3.7}$$

We put

$$\varrho = \frac{1}{2}(R_1 + R_2)\cos 2\alpha + \sqrt{R_1R_2 + \frac{1}{4}(R_1 - R_2)^2\cos^2 2\alpha}$$

Theorem 2. In the case of a bilateral billiard the fixed point (0, 0) of the composition $T_- \circ T_+$ is
(1) for $0 < \alpha < \pi/4$ hyperbolic, if

$$R_1\cos 2\alpha < 2l\cos\alpha < R_2\cos 2\alpha \quad \text{or} \quad 2l\cos\alpha > \varrho$$

elliptic, if

$$0 < 2l\cos\alpha < R_1\cos 2\alpha \quad \text{or} \quad R_2\cos 2\alpha < 2l\cos\alpha < \varrho$$

(2) for $\alpha = \pi/4$ hyperbolic, if

$$\sqrt{2}l > \sqrt{R_1R_2}$$

elliptic, if

$$0 < \sqrt{2}l < \sqrt{R_1R_2}$$

(3) for $\pi/4 < \alpha < \pi/2$ hyperbolic, if

$$2l\cos\alpha > \varrho$$

elliptic, if

$$0 < 2l\cos\alpha < \varrho$$

The proof is based on an analysis of the trace of the product of the linearization matrices (3.7). These conditions become the conditions of stability of the two-link trajectory of the usual billiard as the magnetic field weakens ($R \to \infty$, $\alpha \to 0$) (for inertial motion of the particle) (see [4]).

This research was supported financially by the Russian Foundation for Basic Research (02-01-01059) and the "State Support for Leading Scientific Schools" programme (NSh-136.2003.1).

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Translated by R.C.G.